On “Weak” and “Strong” Population Momentum

Thomas J. Espenshade*
Analia S. Olgiati**
Simon A. Levin***

Working Paper No. 2009-01

June 2009

*Department of Sociology and Office of Population Research, Princeton University

**Woodrow Wilson School of Public and International Affairs and Office of Population Research, Princeton University

***Department of Ecology and Evolutionary Biology, Princeton University

Direct correspondence to: Thomas J. Espenshade, Office of Population Research, 249 Wallace Hall, Princeton University, Princeton NJ 08544; e-mail: tje@princeton.edu; tel. 609-258-5233; fax: 609-258-1039.

An earlier version of this paper was presented at the annual meetings of the Population Association of America, Detroit, MI, April 30-May 2, 2009. Partial support for this research came from the Eunice Kennedy Shriver National Institute of Child Health and Human Development (grant #5R24HD047879). We are grateful to Adrian Banner, Laura Blue, Ronald Brookmeyer, Robert Calderbank, Chang Chung, Erhan Cinlar, Dennis Feehan, David Gabai, Diego Hofman, Igor Klebanov, Edward Nelson, John Palmer, David Potere, Germán Rodríguez, Lily Shen, and Shripad Tuljapurkar for helpful comments and discussions. We thank Chang Young Chung for preparing the figures and Valerie Fitzpatrick for administrative and technical support.

On “Weak” and “Strong” Population Momentum

Abstract

This paper decomposes total population momentum into two constituent and multiplicative parts called “weak” momentum and “strong” momentum. Weak momentum depends on deviations between a population’s observed age distribution and its implied stable age distribution. Strong momentum is a function of deviations between a population’s implied stable and stationary age distributions. In general, the factorization of total momentum into the product of weak and strong momentum is a very good approximation. The factorization is exact, however, if the observed age distribution is stable or if initial fertility is already at replacement. We provide numerical illustrations by calculating weak, strong, and total momentum for 176 countries, the world, and its major regions. In short, the paper brings together disparate strands of the population momentum literature and shows how the various kinds of momentum fit together into a single unifying framework.

INTRODUCTION

The concept of population momentum refers to the fact that a population typically does not stop growing (or declining) the instant its fertility reaches replacement. Instead, in a closed population, growth or decline gradually slows until a stationary population is attained, in much the same way that a car gradually comes to a complete stop after a driver's foot is lifted from the accelerator pedal (Schoen and Kim, 1991). The relative amount of momentum is usually measured by the ratio of the size of the long-run stationary population to that of the population when replacement-fertility is first achieved. Momentum coefficients for individual countries in 1935 ranged from 1.08 for Austria, Belgium, and France to 1.60 for Puerto Rico and Honduras (Vincent, 1945). By 2005 momentum coefficients across all United Nations countries stretched from 0.81 in Bulgaria to 1.76 in Oman (authors' calculations). In several cases, including Georgia, the Netherlands, and Poland, total momentum was essentially zero. These examples illustrate that population momentum can be positive or negative (Knodel, 1999; Preston and Guillot, 1997), and that it can play a large or small role in population dynamics.

One reason for further study of population momentum is that it contributes importantly to future population growth in developing countries. John Bongaarts (1994, 1999) has estimated that momentum accounts for nearly one-half of projected future growth in the developing world over the next century. For the period between 2000 and 2100, momentum is the most important factor in projected future growth for the world and all major regions except Europe and sub-Saharan Africa. For every region in the developing world except sub-Saharan Africa, momentum is a more important contributor to future population growth than all other factors combined (Bongaarts and Bulatao, 1999). During the next half century, momentum is projected to account for 58 percent of future population growth in the developing world (Bongaarts, 2007).

Nor should one overlook the contribution of negative momentum to population decline in the developed world. Even with an immediate fertility rebound back to replacement, Europe's population (ignoring migration) is projected to fall by seven percent from current numbers before leveling off. And, in addition to Bulgaria, 11 other countries (all in Europe except Japan) are projected to decline by more than 10 percent due to negative momentum even if fertility were to rise instantaneously and permanently to replacement.

The purpose of this paper is to convey a deeper understanding of how age composition contributes to population growth or decline. We decompose total population momentum into two constituent and multiplicative parts called "weak" momentum and "strong" momentum. Weak momentum captures deviations between a population's observed age distribution and the stable age distribution implied by current fertility and mortality. Strong momentum reflects deviations between the stable age distribution and the stationary age distribution produced by replacement fertility. In addition, by showing how total, weak, and strong momentum fit together into a single theoretical and empirical framework, the paper unifies a number of disparate and seemingly unrelated concepts in the population momentum literature.[1]

[1] Previous work to decompose total momentum has emphasized its age-specific components (Preston, 1986, 1988; Schoen and Kim, 1991). Guillot (2005) decomposes total momentum into two multiplicative factors: (1) the direct effect of improvements in cohort survivorship and (2) fluctuations in annual numbers of births.

TOTAL, WEAK, AND STRONG MOMENTUM

The stylized ingredients of a unified framework are shown in Figure 1. Let P_0 be the size of any arbitrary initial population—arbitrary with respect to size, age structure, fertility, and mortality. Assume only that the population is closed to migration and consists entirely of females, there are some women younger than the oldest age of childbearing, and fertility and mortality are bounded by the range of contemporary human experience. For the sake of illustration in Figure 1, fertility is above replacement. Suppose that birth rates are lowered instantaneously to replacement at time $t = 0$ by dividing the given fertility schedule by the net reproduction rate (R_0) and that fertility and mortality are then held constant. As shown by the lower solid line, the population eventually converges to a stationary population with size S_0'. The ratio S_0' / P_0 is the usual measure of total population momentum.

[Figure 1 about here]

Suppose instead that the observed population P_0 is projected holding current fertility and mortality constant. It will ultimately converge to a stable population as indicated by the trajectory of the lower dashed line in Figure 1. Once a stable state has been attained, imagine that one uses the stable growth rate to reverse project the size of the stable population back to $t = 0$. The reverse projection follows an exponential curve represented by the upper dashed line. The new population is the stable equivalent population. It has size Q_0, and its age distribution is the stable age distribution implied by the indefinite continuation of current fertility and mortality. It is asymptotically equivalent to the observed population in the sense that if both are projected forward from $t = 0$ holding current fertility and mortality constant, they will eventually become indistinguishable with respect to population size and age composition. We measure weak momentum with the ratio Q_0 / P_0.

Finally, consider a projection of the stable equivalent population using the same constant replacement-level fertility and mortality assumptions used to produce the lower solid line in Figure 1. The size of the population begins at Q_0 and follows the upper solid curve before leveling off at a stationary population of size $S_0^{''}$. Strong momentum is measured by the ratio $S_0^{''} / Q_0$. The stationary populations represented by the endpoints of the two solid lines have the same proportionate age distributions, but they do not necessarily have the same size. We may write total momentum as the identity

$$\text{Total Momentum} = \frac{S_0^{'}}{P_0} \equiv \frac{Q_0}{P_0} \times \frac{S_0^{''}}{Q_0} \times \frac{S_0^{'}}{S_0^{''}} \; .$$

In words, total momentum is the product of weak momentum, strong momentum, and an offset factor represented by the ratio $S_0^{'} \big/ S_0^{''}$. If, however, $S_0^{'} = S_0^{''}$, then

$$\frac{S_0^{'}}{P_0} = \frac{Q_0}{P_0} \times \frac{S_0^{''}}{Q_0} \; , \tag{1}$$

and we would have accomplished an exact factorization of total population momentum into the product of weak and strong momentum. Moreover, under these conditions, the observed population P_0 and its stable equivalent Q_0 are asymptotically equivalent not only with respect to current fertility and mortality but also with respect to replacement fertility and mortality. Our interest will center on the relation between $S_0^{'}$ and $S_0^{''}$ and the conditions under which they might be equal. It is important first to explain how the three types of momentum are related to deviations between pairs of age distributions.

Total Momentum

The concept of the stable equivalent population plays a central and unifying role in our analysis. If an initial arbitrarily chosen population at $t = 0$ has size P_0 with fertility schedule $m(a)$ and survival function $p(a)$, then the stable equivalent population is a stable population whose age distribution and vital rates are determined by $m(a)$ and $p(a)$. The size of the stable equivalent population at $t = 0$ is given by

$$Q_0 = \frac{\int_0^{\beta} n(x)v(x)\,dx}{b \cdot A_r} \quad , \tag{2}$$

where $n(x)dx$ is the number of females between exact ages x and $x + dx$; b and A_r are, respectively, the birth rate and the mean age of childbearing in the stable equivalent population; and β is the oldest age of childbearing. In equation (2), $v(x)$ is Fisher's reproductive value function (Fisher, 1930: 27-30) defined for a woman at exact age x as

$$v(x) = \frac{1}{p(x)} \int_x^{\beta} e^{-r(a-x)} p(a) m(a)\,da \ . \tag{3}$$

One may interpret $v(x)$ as the present discounted value (using the stable growth rate r as the discount rate) of the average expected number of daughters remaining to be born per woman at age x. When the observed population and the stable equivalent population are projected from $t = 0$ holding constant both $m(a)$ and $p(a)$, they will eventually converge and become indistinguishable.

In Figure 1 when the observed population P_0 is projected with fertility set at replacement, it converges to the stationary population whose size is S_0'. In other words,

S_0' is the size of the stable/stationary population that is equivalent to P_0 with respect to *replacement* fertility. We can express this formally, using (2) and (3), as

$$S_0' = \int_0^{\beta} \frac{n(x)}{p(x)} \int_x^{\beta} p(a) m^*(a)\, da\, dx / b^* \cdot A_r^* . \tag{4}$$

Asterisks in (4) are used to indicate values under "replacement" conditions. Assume that $m^*(a)$ is obtained by normalizing the fertility schedule $m(a)$ with the net reproduction rate (R_0); r = 0 by definition. We may rewrite $n(x)$ as $P_0 \cdot c(x)$ where $c(x)$ is the proportionate age distribution in the observed population. And because the stationary population age distribution $c^*(x) = b^* \cdot p(x)$, equation (4) becomes

$$S_0' = \int_0^{\beta} \frac{P_0\, c(x)}{c^*(x)} \int_x^{\beta} p(a) m^*(a)\, da\, dx / A_r^* , \tag{5}$$

from which it follows that

$$\text{Total Momentum} = \frac{S_0'}{P_0} = \int_0^{\beta} \frac{c(x)}{c^*(x)} \int_x^{\beta} p(a) m^*(a)\, da\, dx / A_r^* . \tag{6}$$

An identical expression for total momentum appears in Preston and Guillot (1997: 20-21), and it was anticipated in somewhat different form by Keyfitz (1985: 155-157).[2]

Equation (6) shows that the total momentum contained in a population's age structure depends on the ratio $c(x)/c^*(x)$ reflecting deviations between the observed proportionate age distribution and the stationary proportionate age distribution below the

[2] Vincent (1945) described the phenomenon of population momentum (what he called the "potential increase of a population") and introduced a method to measure it. He also developed the theory behind the stable equivalent population and independently invented Fisher's reproductive value without calling them such. This work was later extended and renamed by Keyfitz (1969, 1971). Another path of development in the momentum literature examines the growth consequences of a gradual decline in fertility to replacement. Pioneering empirical research was undertaken by Frejka (1973). Seminal work by Li and Tuljapurkar (1999, 2000) has sparked a new line of formal analysis.

oldest age of childbearing. In particular, total momentum is a weighted average of these deviations where the age-specific weight is $\int_{x}^{\beta} p(a)m^{*}(a)da / A_{r}^{*}$ (Preston and Guillot, 1997). These weights sum to unity as can be seen by reversing the order of integration in the double integral. Moreover, because the weights are largest prior to the onset of childbearing and then decline toward zero (Preston and Guillot, 1997), deviations between $c(x)$ and $c^{*}(x)$ in the early part of life matter most in determining total momentum. Finally, if the initial population is already stationary so there is no difference between $c(x)$ and $c^{*}(x)$, then all momentum has been wrung out of the age distribution and the momentum coefficient equals 1.0 (Preston and Guillot, 1997).[3]

<u>Weak Momentum</u>

It may be unclear why the ratio Q_0 / P_0 in Figure 1 is a measure of population momentum, especially when most discussions of momentum occur in the context of replacement fertility. Suppose we separate the observed population growth rate at time *t*, $r(t)$, into two parts as

$$r(t) = r + \rho(t) \ , \tag{7}$$

where *r* is the stable growth rate produced by *m*(*a*) and *p*(*a*), and $\rho(t)$ is the residual. Then *r* reflects the contribution of fertility and mortality to $r(t)$, and $\rho(t)$ measures the contribution of the age distribution at time *t* (Espenshade, 1975).

[3] Other work has approximated total momentum by comparing observed and stationary population age distributions. Kim and Schoen (1993) express momentum as the ratio of the proportion in the observed population to the proportion in the stationary population at a given age determined by the cross-over point of their respective reproductive value functions. Momentum has also been approximated by the ratio of the proportion under age 30 in the observed population to the proportion under age 30 in the stationary population (Kim and Schoen, 1997; Kim, Schoen, and Sarma, 1991).

If P_0 is projected with constant fertility and mortality, its size at time T will be

$$P_T = P_0\, e^{\int_0^T r(t)dt} . \tag{8}$$

Because the size of its stable equivalent population at t = 0, Q_0 , may be found by reverse projection,

$$Q_0 = \lim_{T\to\infty}\left\{ P_0\, e^{\int_0^T r(t)dt} / e^{rT} \right\} . \tag{9}$$

Substituting (7) into (9) yields

$$Q_0 = P_0 \lim_{T\to\infty}\left\{ e^{\int_0^T \rho(t)dt} \right\} . \tag{10}$$

The limit in (10) exists because $\rho(t)$ is the transient part of $r(t)$, and $\rho(t) \to 0$ as the observed population converges to a stable population. Equation (10) tells us how large P_0 would eventually become if age composition were the only source of population growth or decline.

Dividing (10) by P_0 produces

$$\frac{Q_0}{P_0} = \lim_{\mathrm{T}\to\infty}\left\{ e^{\int_0^T \rho(t)dt} \right\} . \tag{11}$$

The ratio Q_0 / P_0 is the relative change in population size due to weak momentum if given fertility and mortality are held constant until a stable population is attained (Espenshade and Campbell, 1977). We call it “weak” momentum because it is based on current fertility and does not require the “strong” assumption that replacement fertility is adopted or imposed.

Weak momentum can be expressed in a form similar to equation (6). To find the size of the stable equivalent population, Q_0, that corresponds with P_0, use (2) and (3) in conjunction with *m*(*a*), *p*(*a*), and the observed age distribution *n*(*x*). It follows that

$$Q_0 = P_0 \int_0^\beta \frac{n(x)}{P_0} \cdot \frac{1}{be^{-rx} p(x)} \int_x^\beta e^{-ra} p(a) m(a) \, da \, dx / A_r \, . \tag{12}$$

By writing $n(x)/P_0$ as $c(x)$ and recognizing that the proportionate stable age distribution $c_r(x) = be^{-rx} p(x)$, equation (12) becomes

$$\text{Weak Momentum} = \frac{Q_0}{P_0} = \int_0^\beta \frac{c(x)}{c_r(x)} \int_x^\beta e^{-ra} p(a) m(a) \, da \, dx / A_r \, . \tag{13}$$

Equation (13) says that weak momentum is a function of deviations between the observed and the implied stable proportionate age distributions below the oldest age of childbearing. If the initial population is already stable so there are no deviations, then $c(x) = c_r(x)$ at all ages, there is no weak momentum, and $Q_0 / P_0 = 1$ as can be verified by reversing the order of integration in (13). The same process shows that weak momentum is a weighted average of disparities between $c(x)$ and $c_r(x)$ where the weights are $\int_x^\beta e^{-ra} p(a) m(a) \, da / A_r$ and sum to one. Equation (13) gives equal weight to deviations up to the age when childbearing begins and monotonically declining weight thereafter until the weight becomes zero at the highest age of childbearing.

The ratio Q_0 / P_0 appears in other places in the momentum literature. Bourgeois-Pichat (1971) introduced the concept of the "inertia of a population" and developed the "coefficient of inertia" (Q_0 / P_0) to measure it. He went on to suggest a further decomposition of weak momentum into the effects of fertility and mortality on the one hand and those of age distribution on the other. In reformulating the concept of the stable equivalent population, Keyfitz (1969) calculated Q_0 / P_0 for several empirical examples without relating it to population momentum. The closest he comes is referring to Q_0 as "a simple measure of the favorability of the age distribution to reproduction, given the current regime of mortality and fertility" (Keyfitz, 1969: 264). Schoen and Kim (1991: 456) have argued, "The momentum concept need not be limited to cases in which the ultimate stable population has zero growth. More generally, momentum can be defined as the size of a population relative to the size of its stable equivalent." Finally, Feeney (2003: 648) has claimed, "The *momentum* of the given age distribution with respect to the *given* age schedules of fertility and mortality is the ratio [Q_0 / P_0]" (first emphasis in the original; second emphasis added).

<u>Strong Momentum</u>

The last kind of momentum shown in Figure 1 is "strong" momentum, defined by the ratio $S_0^{''} / Q_0$. Because P_0 and its proportionate age distribution, $c(x)$, can be chosen arbitrarily, let the starting population (that is, the initial observed population) be stable with size Q_0 and age distribution $c_r(x) = be^{-rx} p(x)$. Then set fertility at replacement and project until the population becomes stationary with growth rate r = 0 and fixed size $S_0^{''}$. The formula in (6) for total momentum may be invoked and applied in this new situation to yield an expression for strong momentum as

$$\text{Strong Momentum} = \frac{S_0^{''}}{Q_0} = \int_0^\beta \frac{c_r(x)}{c^*(x)} \int_x^\beta p(a) m^*(a) \, da \, dx / A_r^* . \qquad (14)$$

Once again, the amount of momentum depends upon deviations between pairs of proportionate age distributions. In the case of strong momentum, what matters is the deviation between the stable and the stationary age distributions, weighted by the same age-specific weights used in (6). These weights add to one and are constant prior to the onset of childbearing and then decline steadily to zero by age β. If fertility is already at replacement in the observed population, then the stable age distribution will be stationary, $c_r(x) = c^*(x)$ at all ages, $S_0^{''}/Q_0 = 1$, and there will be no strong momentum.

Strong momentum is identical to what we might call "Keyfitz" momentum. Keyfitz (1971) considered the subsequent growth in an initially *stable* population if fertility rates were set immediately to replacement by normalizing the fertility schedule, $m(a)$, with the net reproduction rate (R_0). To see the equivalence between (14) and the Keyfitz formula, rewrite (14) as

$$\frac{S_0^{''}}{Q_0} = \int_0^\beta \frac{be^{-rx}p(x)}{b^* p(x)} \int_x^\beta p(a)m(a)\,da\,dx \,/\, A_r^* \cdot R_0 \ , \tag{15}$$

where b is the birth rate in the initial stable population, and b^* is the birth rate in the ultimate stationary population and equal to the reciprocal of life expectancy at birth (e_0^o). Simplifying (15) and reversing the order of integration, we have

$$\frac{S_0^{''}}{Q_0} = \frac{be_0^o}{A_r^* R_0} \int_0^\beta \int_0^a e^{-rx} p(a)m(a)\,dx\,da \ . \tag{16}$$

But because

$$\int_0^a e^{-rx}dx = -\frac{1}{r}\int_0^a e^{-rx}(-r)\,dx = \frac{1}{r}\left(1-e^{-ra}\right) \ ,$$

(16) becomes

$$\frac{S_0^{''}}{Q_0} = \frac{be_0^o}{rA_r^* R_0}\left(R_0 - 1\right) \ , \tag{17}$$

which is exactly the expression in Keyfitz (1971: 76) for momentum in a stable population.

In related work, Kim and Schoen (1993) have derived momentum in an initially stable population as a simple ratio of proportionate stable and stationary age distributions at a given age determined by the crossover point of reproductive value functions. A further decomposition of momentum in a stable population is given in Kim and Schoen (1997). Much of the work on momentum in populations with gradually declining fertility assumes initial stability. Schoen and Jonsson (2003) decompose this momentum into one part that reflects the effect of growth continuing at the initial stable rate for half the period of decline and an offsetting factor that reduces momentum and the number of births in the long-run stationary population due to population aging (if fertility is falling). Finally, Goldstein and Stecklov (2002) develop a simple analytic formula for estimating population momentum in a stable population when fertility declines gradually and linearly to replacement. Their analysis suggests that differences between the observed population and its stable equivalent are sufficiently small that they can be ignored (Goldstein and Stecklov, 2002: 136-137). In other words, their paper assumes that weak momentum is inconsequential to an understanding of total momentum and only strong momentum matters for all practical purposes.

Examples

To add empirical content to our analysis and anticipate later results, we show projections of the female populations for two countries in Figure 2. The top portion illustrates the case of Indonesia, a population with above-replacement fertility. Females numbered 113.1 million in 2005, and the size of the stable equivalent population is 139.7

million. When both populations are projected assuming replacement fertility, it is not possible to detect any visual distance between $S_0^{'}$ and $S_0^{''}$. They are both very close to 150 million. A similar situation arises in the case of Japanese women shown in the bottom portion of Figure 2. Here the example is chosen to reflect fertility below replacement. But once again, it appears that the observed population and its stable equivalent are asymptotically equivalent with respect to both current and replacement fertility. In this example, $S_0^{'}$ and $S_0^{''}$ are approximately 57.5 million. For Indonesian and Japanese women in 2005, the data in Figure 2 suggest that total momentum is exactly the product of weak and strong momentum, or very nearly so. In the next section, we investigate analytically whether $S_0^{'} = S_0^{''}$ in all situations.

[Figure 2 about here]

A UNIFIED ANALYTIC FRAMEWORK

Because population projections usually rely on discrete formulations of age and time, it is convenient to develop an analytic solution using matrix algebra. Relevant introductions are contained in Finkbeiner (1960: 23-43) and Keyfitz (1968: 27-73).

The General Case

Let $\left\{\mathbf{P}^{(0)}\right\}$ be an $n \times 1$ vector for the age distribution of an arbitrarily chosen population at time t = 0. The elements in the vector represent the number of females in successive discrete (say, five-year) age intervals. Suppose that current fertility and mortality are captured in a standard $n \times n$ Leslie matrix $\mathbf{L}$. Let the Leslie matrix $\mathbf{L}_*$

reflect replacement fertility, and assume that $\mathbf{L}_*$ is obtained from $\mathbf{L}$ by normalizing fertility rates by the net reproduction rate (R_0).[4] Then we may write

$$\left\{\mathbf{P}^{(0)}\right\} = \left\{\mathbf{Q}^{(0)}\right\} + \left\{\mathbf{V}^{(0)}\right\}, \tag{18}$$

where $\left\{\mathbf{Q}^{(0)}\right\}$ is the age distribution of the stable equivalent population, and $\left\{\mathbf{V}^{(0)}\right\}$ is the deviation of $\left\{\mathbf{P}^{(0)}\right\}$ from $\left\{\mathbf{Q}^{(0)}\right\}$. When (18) is projected forward infinitely with $\mathbf{L}$, the part associated with $\left\{\mathbf{V}^{(0)}\right\}$ disappears.

Next consider the infinite projection of (18) using replacement fertility,

$$\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{P}^{(0)}\right\} = \lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{Q}^{(0)}\right\} + \lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{V}^{(0)}\right\}. \tag{19}$$

But because $\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{P}^{(0)}\right\}$ is the stationary population vector $\left\{\mathbf{S}'\right\}$ and $\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{Q}^{(0)}\right\}$ is the stationary population vector $\left\{\mathbf{S}''\right\}$ with the same proportionate age distribution as $\left\{\mathbf{S}'\right\}$, we have from (19) the quantity of interest, which is

$$\left\{\mathbf{S}'\right\} - \left\{\mathbf{S}''\right\} = \lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{V}^{(0)}\right\}. \tag{20}$$

Now change bases and rewrite $\left\{\mathbf{V}^{(0)}\right\}$ as a linear combination of the eigenvectors $\left\{\mathbf{Z}_j\right\}$ of $\mathbf{L}_*$, specifically as

$$\left\{\mathbf{V}^{(0)}\right\} = \mathrm{k}_1\left\{\mathbf{Z}_1\right\} + \mathrm{k}_2\left\{\mathbf{Z}_2\right\} + \cdots + \mathrm{k}_\mathrm{n}\left\{\mathbf{Z}_\mathrm{n}\right\},$$

or more compactly as

$$\left\{\mathbf{V}^{(0)}\right\} = \mathrm{k}_1\left\{\mathbf{Z}_1\right\} + \left\{\mathbf{W}^{(0)}\right\}. \tag{21}$$

---

[4] More formally, $\mathbf{L}_* = \mathbf{I}_*\,\mathbf{L}$, where $\mathbf{I}_*$ is an $\mathrm{n}\times\mathrm{n}$ identity matrix except that the element in the first row and column of $\mathbf{I}_*$ is $1/R_0$ and not 1.

In (21), $\mathrm{k}_1\{\mathbf{Z}_1\}$ is the stationary part of $\{\mathbf{V}^{(0)}\}$. In particular, $\mathrm{k}_1\{\mathbf{Z}_1\}$ is the stationary equivalent population, where k_1 is the stationary population size and $\{\mathbf{Z}_1\}$ is the proportionate stationary age distribution. $\{\mathbf{Z}_1\}$ is also the principal eigenvector of $\mathbf{L}_*$ associated with the principal or dominant eigenvalue $\lambda_1 = 1$.

Finally, consider the infinite projection of (21) using $\mathbf{L}_*$, whereupon

$$\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{V}^{(0)}\right\} = \lim_{t\to\infty}\mathbf{L}_*^t\ \mathrm{k}_1\{\mathbf{Z}_1\} + \lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{W}^{(0)}\right\}. \tag{22}$$

The second term on the right-hand side of (22) disappears because $\{\mathbf{V}^{(0)}\}$ and $\mathrm{k}_1\{\mathbf{Z}_1\}$ are asymptotically equivalent under $\mathbf{L}_*$. Moreover, $\mathbf{L}_*\ \mathrm{k}_1\{\mathbf{Z}_1\} = \mathrm{k}_1\{\mathbf{Z}_1\}$. That this is true is immediately obvious from demography, because projecting an already stationary population using replacement-level fertility simply reproduces the original population. In addition, however, because $\{\mathbf{Z}_1\}$ is an eigenvector of $\mathbf{L}_*$,

$$\mathbf{L}_*\ \mathrm{k}_1\{\mathbf{Z}_1\} = \mathrm{k}_1\ \mathbf{L}_*\{\mathbf{Z}_1\} = \mathrm{k}_1\ \lambda_1\{\mathbf{Z}_1\} = \mathrm{k}_1\{\mathbf{Z}_1\}.$$

It follows from (22) that $\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{V}^{(0)}\right\} = \mathrm{k}_1\{\mathbf{Z}_1\} + \{\mathbf{0}\}$ and, therefore, from (20) that

$$\left\{\mathbf{S}'\right\} - \left\{\mathbf{S}''\right\} = \mathrm{k}_1\{\mathbf{Z}_1\}. \tag{23}$$

In general, because $\mathrm{k}_1 \neq 0$, it follows that, in general, $\left\{\mathbf{S}'\right\} \neq \left\{\mathbf{S}''\right\}$. We conclude that when an observed population and its stable equivalent are projected on the assumption of replacement-level fertility, the stationary populations to which they converge are usually different. In other words, the factorization of total momentum into the product of weak momentum and strong momentum typically is not exact, but only approximate.

Two Special Cases

There are, however, two special cases in which the factorization is exact.

(i) Initial Stability. Suppose the initial age distribution is stable with respect to current fertility and mortality. Then $\left\{\mathbf{P}^{(0)}\right\}=\left\{\mathbf{Q}^{(0)}\right\}$, and $\left\{\mathbf{V}^{(0)}\right\}=\{\mathbf{0}\}$. Under this condition, from (20), $\left\{\mathbf{S}'\right\}=\left\{\mathbf{S}''\right\}$. In this circumstance, there is no weak momentum, and total momentum consists entirely of strong momentum. Developing countries are more likely than developed countries to be covered by this situation.

(ii) $R_0 = 1$. If current fertility is at replacement so that $\mathbf{L}=\mathbf{L}_*$, we may rewrite (18) as $\left\{\mathbf{P}^{(0)}\right\}=\left\{\mathbf{Q}_*^{(0)}\right\}+\left\{\mathbf{V}^{(0)}\right\}$, where $\left\{\mathbf{Q}_*^{(0)}\right\}$ is the stable/stationary equivalent population. Under infinite projection by current fertility and mortality, $\left\{\mathbf{P}^{(0)}\right\}$ and $\left\{\mathbf{Q}_*^{(0)}\right\}$ are asymptotically equivalent so that $\lim_{t\to\infty}\mathbf{L}_*^t\left\{\mathbf{V}^{(0)}\right\}=\{\mathbf{0}\}$. Once again, by (20), $\left\{\mathbf{S}'\right\}=\left\{\mathbf{S}''\right\}$. When initial fertility is at replacement, there is no strong momentum. All momentum is weak momentum. Fertility at or near replacement level is more likely to characterize developed countries than developing ones.

There is a final degenerate case that is worth mentioning in passing. If fertility is initially at replacement and the initial age distribution is stationary, then there is no population momentum whatsoever. Neither is there any strong or weak momentum. Total momentum factors identically into the product of strong and weak momentum because each of the ratios in equation (1) is 1.0.

Evaluating the Approximation

Our analysis has shown that apart from two special cases, the sizes of the ultimate stationary populations in Figure 1, $S_0^{'}$ and $S_0^{''}$, are generally not identical. But how close are they in practice? If the deviations between $S_0^{'}$ and $S_0^{''}$ are small, then the factorization of total momentum into the product of weak and strong momentum can be a useful approximation, even if it does not always hold exactly.[5] We investigated the relationship between $S_0^{'}$ and $S_0^{''}$ for each of 176 United Nations countries. Projections were made using a standard cohort-component methodology applied to the most recent United Nations (2007) data. The baseline population comes from an estimate for July 1, 2005 for females, disaggregated by five-year age groups from 0-4 up to 100 years and older. Fertility rates and sex ratios at birth are based on estimates for the period 2000-2005, and projections that assume replacement fertility are constructed by dividing age-specific fertility rates by the net reproduction rate. Estimates of death rates by five-year age groups up to 100 years and older come from the World Health Organization (2008). All projections are carried out for 300 years assuming a closed population.[6]

We have graphed on a logarithmic scale in Figure 3 values of $S_0^{'}$ and $S_0^{''}$ for the female populations of 176 U.N. countries. Values of $S_0^{'}$ range from 73 million for Tonga and 77 million for Grenada to 736 million for China and 777 million for India. But the most striking feature of Figure 3 is that all the points appear to lie on the 45-degree line.

---

[5] One is reminded here of a comment by Samuel Preston (1988: 495) that, "Approximations are not to be judged by whether they are exact and error-free, but by how well they function as approximations."

[6] Nineteen United Nations countries are not WHO members, and we have excluded them from our analysis. They range in size from Aruba (total population of 103,000 including men and women) to Hong Kong (with a population of 7.1 million). The average population size of the excluded countries is approximately 1.1 million total persons.

If $S_0^{'}$ and $S_0^{''}$ are not equal to one another, then the deviation between them is very small. The simple correlation coefficient between $S_0^{'}$ and $S_0^{''}$ is 0.9999. The correlation is identical to four decimals when population size is measured on a logarithmic scale in base 10. The data in Figure 3 reinforce an important conclusion. In cases where the factorization of total momentum into the product of weak and strong momentum is not exact, the approximation is extremely good.

[Figure 3 about here]

Figure 4 examines the relationship between $S_0^{'}$ and $S_0^{''}$ in another way. Here we show the distribution of the percentage deviation of $S_0^{''}$ from $S_0^{'}$ for the same 176 countries. Most of the deviations cluster in a tight pattern around zero, and only a small number fall outside the range of ± 0.5 percent. Roughly two out of every five cases (39.2 percent) fall within 0.1 percent of the origin. In two-thirds of the cases (64.8 percent), the deviations are contained within 0.2 percent. And in three-fourths of all cases (74.4 percent), the relative difference between $S_0^{'}$ and $S_0^{''}$ lies within 0.3 percent.[7]

[Figure 4 about here]

The data in Figures 3 and 4 point to one overarching conclusion. When $S_0^{'}$ and $S_0^{''}$ are compared, only one of two outcomes is possible. Either $S_0^{'} = S_0^{''}$ or $S_0^{'} \approx S_0^{''}$. This means that total momentum is either identically equal to the product of weak and strong momentum or very close to it. And we now know why this is the case. Because for many of the world's countries, either current fertility is close to replacement or the age distribution is nearly stable. It is only in instances of a joint departure from replacement

[7] The percentage deviations in Figure 4, including all 176 countries, have a standard deviation of 0.331, a mean value of 0.033, and a median value of 0.047.

fertility *and* age distribution stability that the exact factorization begins to dissolve into an approximation.

MOMENTUM IN GLOBAL PERSPECTIVE

Using concepts developed in conjunction with Figure 1, we calculated values for total, weak, and strong momentum for each of 176 United Nations countries, broad regional aggregates, and the world. Results for the world and its major regions are reported in Table 1. Notice first that there is excellent agreement between the numbers in columns 1 and 4. Any differences are usually limited to the third decimal place, which suggests that the product of weak and strong momentum is an unusually good approximation to total momentum. Moreover, our estimates indicate that world population would grow by an additional 40 percent if global fertility rates had moved instantaneously to replacement in 2005. Weak and strong momentum contribute roughly equal shares to world population momentum. Taking natural logarithms shows that weak momentum accounts for about 53 percent of total world momentum and strong momentum contributes roughly 47 percent.

[Table 1 about here]

Europe and the least developed countries represent two extremes on the global momentum scale. The least developed countries possess the largest values for total and strong momentum in Table 1 and have one of the lowest values for weak momentum. On the contrary, Europe has the lowest values for total and strong momentum but the largest value for weak momentum. Because total momentum is a function of the ratio between proportions in the observed population and the stationary population at young ages, it will be influenced by the recent history of crude birth rates. High rates induce large values for total momentum; low birth rates predict low values for total momentum. Europe's crude birth rate was 10.2 per 1,000 for 2000-2005, in contrast to a birth rate of 37.6 per 1,000 for the least developed countries during the same period (United Nations, 2007). Data in Table 1 suggest that if replacement fertility had been adopted in 2005 and remained constant, the population of the least developed countries would eventually grow by more than one-half (51.3 percent) before becoming stationary. Even with an increase in fertility to replacement in 2005, Europe's population would ultimately decline by seven percent. The only other region exhibiting negative total momentum is the group of more developed countries. Several regions have positive total momentum coefficients in the neighborhood of 1.50.

Weak momentum depends on relative proportions in the observed and stable age distributions before the onset of childbearing. Populations whose fertility is substantially below replacement and whose age distributions have not had time to adjust fully to the new fertility regime will tend to have high values for weak momentum. With a net reproduction rate of 0.69 for 2000-2005, Europe has the largest value in Table 1 for weak momentum (1.383). On the other hand, populations with high and relatively constant fertility will have age distributions that are approximately stable. This condition produces weak momentum coefficients near unity. The least developed countries, sub-Saharan Africa, and Africa as a whole all have weak momentum values close to 1.0. And each has a history of high fertility with modest declines occurring only recently (United Nations, 2007).

Values for strong momentum involve a comparison between a population's stable and stationary age distributions in the earliest part of life. When fertility is substantially above replacement, the stable age distribution will be young relative to its stationary counterpart. Coefficients of strong momentum should be large in this situation. But if fertility is dramatically below replacement, the opposite circumstance will arise and values for strong momentum will be less than one. As seen in Table 1, strong momentum is greatest (at or above 1.45) in regions with high fertility. Europe's coefficient of strong momentum is just 0.67—the lowest of any region.

Notice finally in Table 1 that the same value for total momentum can be produced with different combinations of weak and strong momentum. The total momentum coefficient is either 1.49 or 1.50 for Latin America and the Caribbean, less developed regions excluding China, and all of Africa. Weak momentum is larger than strong momentum in Latin America. These roles are reversed for less developed regions except China. And strong momentum accounts for practically all of total momentum in Africa.

Table 1 has already suggested that results depend on the level of development. To clarify this relationship further, Figure 5 shows the distribution of values for total momentum for a group of 43 more developed countries, 133 less developed countries, and for all countries. For more developed countries in panel (a), modal values fall between 0.9 and 1.0, indicating that negative population momentum is not uncommon in richer countries. Total momentum values for less developed countries in panel (b) cluster near 1.5. The distribution in panel (c) for all countries exhibits a somewhat bimodal shape, but is weighted toward values near 1.5 because of the greater number of less developed countries. For added perspective, panel (c) also contains a vertical line to indicate total momentum for the entire world.[8]

[8] Mean and median values, respectively, in Figure 5 are as follows: in panel (a), 0.9721 and 0.9610; in panel (b), 1.4514 and 1.4910; and in panel (c), 1.3343 and 1.4285.

[Figure 5 about here]

Finally we consider weak and strong momentum values for individual countries. Figure 6 contains a scatter plot of points whose coordinates correspond to strong and weak momentum for 176 countries. Strong momentum ranges between 0.48 and 1.75. Countries with the largest values include Timor-Leste (1.751), Guatemala (1.633), Yemen (1.580), Madagascar (1.574), and Guinea-Bissau (1.571). The five smallest values belong to Ukraine (0.479), Czech Republic (0.530), Bulgaria (0.535), Belarus (0.541), and Slovakia (0.546). The level of the net reproduction rate (R_0) is an important determinant of strong momentum. In the five countries where strong momentum is greatest, R_0 -values are 2.1 or higher. By contrast, R_0 -values do not exceed 0.60 among the five countries with the smallest coefficients for strong momentum.

[Figure 6 about here]

Weak momentum varies between 0.96 and 1.97. Countries with the smallest values include Sierra Leone (0.961), Timor-Leste (0.965), Guinea-Bissau (0.969), Mozambique (0.970), and Malawi (0.974). A history of high and relatively constant fertility is a good predictor of weak momentum near 1.0. Each of these countries has a total fertility rate above 5.5 for the period 2000-2005 (United Nations, 2007). On the other hand, fertility that has fallen recently to low levels is indicative of large values for weak momentum. Countries with the highest values include Republic of Korea (1.970), Armenia (1.918), Slovakia (1.824), Poland (1.758), and Azerbaijan (1.754). Fertility experienced a recent collapse in each of these cases (United Nations, 2007).[9]

[9] Low fertility by itself is not enough to produce large values for weak momentum. Italy's total fertility rate in 2000-2005 was 1.29 (United Nations, 2007), similar to that in Korea (1.24), Armenia (1.35), Slovakia (1.22), and Poland (1.25). But Italy's weak momentum coefficient is only 1.382, because Italian

Total momentum values of 1.0 and 1.5, respectively, are shown by points along the two hyperbolic curves in Figure 6. It is clear from the graph that multiple combinations of strong and weak momentum are compatible with the same total momentum. Most countries lie near or between the two curves, as panel (c) in Figure 5 suggests they should. Oman has the largest total momentum (1.760), followed by Nicaragua (1.737), Guatemala (1.732), and Honduras (1.703). In each case, population is expected to grow by more than 70 percent if replacement fertility had been adopted in 2005. The lowest values for total momentum belong to Bulgaria (0.811), Ukraine (0.824), Germany (0.844), and Italy (0.845). For these countries, even if fertility rebounded immediately to replacement, population losses between 15 and 20 percent could be expected.

Countries in Figure 6 are distinguished by their level of development. There are some overlapping circles, squares, and crosses in the graph, and the respective scatter plots for MDCs and LDCs are not totally distinct. But in general, the least developed countries are distributed around the 1.5 total momentum curve, and more developed countries are located near or somewhat below the 1.0 curve. That there are so many points beneath the 1.0 momentum curve emphasizes how important negative population momentum is among more developed countries (Preston and Guillot, 1997). Points for other less developed countries are more dispersed between the two total momentum curves. If one imagines a 45-degree line drawn from the origin, all of the more developed countries lie above this line. For every MDC, weak momentum is greater than

fertility has been low for several decades, giving the younger part of its age distribution time to adjust to lower fertility.

strong momentum.[10] Most, although certainly not all, LDCs appear to lie below the 45-degree line, meaning that strong momentum outweighs weak momentum. The point for the World is indicated by a shaded triangle. This point lies on or close to the 45-degree line, which suggests in another way that on a global scale, weak and strong momentum are roughly equal in magnitude. The takeaway message is this: as the level of development increases in Figure 6, fertility levels generally fall, overall population momentum is less, strong momentum becomes weaker, and weak momentum becomes stronger.

DISCUSSION

This paper has described a decomposition of overall population momentum into two constituent and multiplicative parts called weak momentum and strong momentum. Weak momentum refers to population change imparted solely by undulations in age composition as a population converges to stability, holding current fertility and mortality constant. The annual rate of change is governed by differences between observed and stable rates of population growth. Weak momentum can also be expressed as a weighted average of age-specific deviations between a population's observed and corresponding stable age distributions. In this sense, weak momentum reflects recent trends in fertility. Strong momentum captures the additional amount of population change in an initially stable population when fertility goes immediately to replacement. The concept of strong momentum is the same as "Keyfitz" momentum. We express it as a weighted average of age-related deviations between a population's stable and stationary age distributions. The value for strong momentum is dictated by a population's current level of fertility in

[10] In addition, every MDC has a strong momentum value less than 1.0, which coincides with below-replacement fertility. The lone exception is Albania whose net reproduction rate is 1.05.

relation to mortality. For both weak and strong momentum, a momentum coefficient equal to 1.0 means there is no momentum of that kind.

As others have demonstrated, total population momentum depends on deviations between a population's observed and stationary age distributions. Our paper shows analytically that the product of weak and strong momentum is not in general identically equal to total momentum, except in two special cases. If current fertility is already at replacement or if the initial age distribution is already stable, then total momentum factors exactly into weak momentum times strong momentum. However, when we calculate values for total, weak, and strong momentum for 176 individual countries, the world, and its major regions, the empirical work shows that the product of weak and strong momentum is an extremely good approximation to total momentum when neither of the special cases holds. The reason is that the two special cases approximate much of contemporary human experience around the world. Fertility is near replacement in a number of more developed countries, while age distributions are not too far from stability in many less developed countries.

Values for total population momentum cluster around 1.5 for less developed countries and around 1.0 for more developed countries. Numerous countries in the latter category exhibit negative population momentum. Europe and the least developed countries represent the two extremes on a global momentum scale. Europe has the lowest value for total momentum (0.93) and the least developed countries as a group have the highest value (1.51). More important, when individual components of total momentum are considered separately, Europe's coefficient of strong momentum is the lowest anywhere, and the highest value belongs to the least developed countries. These roles are reversed for weak momentum. Europe's value is largest, and the least developed countries possess one of the smallest values. Among individual countries, as the level of development increases, fertility tends to decline, values for total momentum fall, strong momentum becomes weaker, and weak momentum becomes stronger. When viewed as a whole, the paper unites disparate strands of the population momentum literature and shows how the various kinds of momentum researchers have considered fit together into a single analytic and empirical framework.

At the same time, the paper raises a number of questions. First, we have identified two special cases in which the factorization of total momentum is exact. But are there more than two? If so, these additional examples might help to explain why the simple product of weak and strong momentum is such a good approximation to total momentum.

Second, the factorization of total momentum is inexact whenever $k_1 \neq 0$ in equation (23). This corresponds to situations in Figure 1 in which $S_0^{'} \neq S_0^{''}$. But we lack a good understanding of conditions that determine the magnitude and direction of k_1.

Third, work by Li and Tuljapurkar (1999, 2000) has opened up new avenues of inquiry concerning momentum in populations with gradually declining fertility.[11] Might it be useful in this context to contemplate the roles of weak and strong momentum?

Fourth, how does our understanding of weak and strong momentum change if populations are no longer assumed to be closed to migration? Migrants modify a population's age distribution (Guillot, 2005: 293), but they can also affect levels of fertility and mortality.

[11] For examples, see Goldstein (2002), Goldstein and Stecklov (2002), O'Neill, Scherbov, and Lutz (1999), Schoen and Jonsson (2003), and Schoen and Kim (1998).

Fifth, our analysis has been largely static, relying on estimates of momentum in 2005 for individual countries, the world, and major regions. There is a need to put weak and strong momentum in a more dynamic context and examine what happens within populations over time. In Figure 6, for example, European countries are clustered in the upper left-hand corner of the diagram. But if one assumes that fertility and mortality as measured in 2005 are held constant, one can imagine the circles for European countries gradually drifting downward toward a weak momentum value close to 1.0 if weak and strong momentum are recalculated every five years after 2005. More broadly, it would be useful to trace the arc of weak and strong momentum across the demographic transition. From this dynamic perspective, one could conjecture that weak momentum is something that emerges during a time of fertility transition and then begins to disappear after birth rates reach relatively low levels. In this sense, weak momentum might be considered the transitory part of total population momentum.

Finally, different combinations of weak and strong momentum are compatible with the same amount of total population momentum. For example, in 2005 total momentum measured 1.46 in Chad and 1.45 in Mongolia. Both countries could be expected to grow by nearly 50 percent if fertility had gone immediately to replacement. In Chad strong momentum (1.49) outweighed weak momentum (0.98). But these roles were reversed in Mongolia where strong and weak momentum were estimated at 0.96 and 1.51, respectively. Apart from the implications their respective life expectancies have for the ultimate stationary population age distribution, does the particular mix of weak and strong momentum in the two countries matter for the paths taken to stationarity?[12] Does

[12] In 2005 female life expectancy at birth was estimated at 52.0 years in Chad compared with 68.4 years in Mongolia. Chad had a total fertility rate of 6.54, and 18.6 percent of all persons were under age 5. By

the time required to achieve a stationary population depend on the relative strength of weak and strong momentum? What about the ripples that are created in various age groups along the way? Carlson (2008) has shown for the United States that successive generations measured by 20-year birth cohorts not only varied in size, but also that there have been substantial booms and busts in their relative sizes. These oscillations have impacts throughout the life course, affecting schooling, labor and housing markets, political attitudes and participation, health care, and pension systems. For given levels of total momentum, what are the corresponding implications of varying combinations of weak and strong momentum as cohorts of fluctuating sizes pulse through institutional arteries?

contrast, the total fertility rate for Mongolia was 2.07, and just 9.1 percent of the population were under age 5 (United Nations, 2007).

References

Bongaarts, John. 1994. “Population Policy Options in the Developing World,” *Science*, February 11, 263(5148): 771-776.

Bongaarts, John. 1999. “Population Momentum.” Pp. 3-15 in Andrew Mason, Thomas Merrick, and R. Paul Shaw, eds., *Population Economics, Demographic Transition, and Development: Research and Policy Implications*. World Bank Working Paper, Washington, DC: IBRD/World Bank.

Bongaarts, John. 2007. “Population Growth and Policy Options in the Developing World.” Paper prepared for the Beijing Forum, Beijing, November 1-4. New York: The Population Council.

Bongaarts, John and Rodolfo A. Bulatao. 1999. “Completing the Demographic Transition,” *Population and Development Review*, September, 25(3): 515-529.

Bourgeois-Pichat, Jean. 1971. “Stable, Semi-Stable Populations and Growth Potential,” *Population Studies*, July, 25(2): 235-254.

Carlson, Elwood. 2008. *The Lucky Few: Between the Greatest Generation and the Baby Boom*. New York: Springer Publishers.

Espenshade, Thomas J. 1975. “The Stable Decomposition of the Rate of Natural Increase,” *Theoretical Population Biology*, August, 8(1): 97-115.

Espenshade, Thomas J. and Gregory Campbell. 1977. “The Stable Equivalent Population, Age Composition, and Fisher’s Reproductive Value Function,” *Demography*, February, 14(1): 77-86.

Feeney, Griffith. 2003. “Momentum of Population Growth.” Pp. 646-649 in Paul Demeny and Geoffrey McNicoll, eds., *Encyclopedia of Population*, Volume 2. New York: Macmillan Reference USA.

Finkbeiner, Daniel T. 1960. *Introduction to Matrices and Linear Transformations*. San Francisco: W.H. Freeman and Company.

Fisher, Ronald A. 1930. *The Genetical Theory of Natural Selection*. Oxford: The Clarendon Press.

Frejka, Tomas. 1973. *The Future of Population Growth: Alternative Paths to Equilibrium*. New York: John Wiley & Sons.

Goldstein, Joshua R. 2002. “Population Momentum for Gradual Demographic Transitions: An Alternative Approach,” *Demography*, February, 39(1): 65-73.

Goldstein, Joshua R. and Guy Stecklov. 2002. “Long-Range Population Projections Made Simple,” *Population and Development Review*, March, 28(1): 121-141.

Guillot, Michel. 2005. “The Momentum of Mortality Change,” *Population Studies*, November, 59(3): 283-294.

Keyfitz, Nathan. 1968. *Introduction to the Mathematics of Population*. Reading, MA: Addison-Wesley Publishing Company.

Keyfitz, Nathan. 1969. “Age Distribution and the Stable Equivalent,” *Demography*, August, 6(3): 261-269.

Keyfitz, Nathan. 1971. “On the Momentum of Population Growth,” *Demography*, February, 8(1): 71-80.

Keyfitz, Nathan. 1985. *Applied Mathematical Demography*. Second Edition. New York: Springer-Verlag.

Kim, Young J. and Robert Schoen. 1993. “Crossovers that Link Populations with the Same Vital Rates,” *Mathematical Population Studies*, 4(1): 1-19.

Kim, Young J. and Robert Schoen. 1997. “Population Momentum Expresses Population Aging,” *Demography*, August, 34(3): 421-427.

Kim, Young J., Robert Schoen, and P. Sankara Sarma. 1991. “Momentum and the Growth-Free Segment of a Population,” *Demography*, February, 28(1): 159-173.

Knodel, John. 1999. “Deconstructing Population Momentum,” *Population Today*, March, Washington, DC: Population Reference Bureau, Inc.

Li, Nan and Shripad Tuljapurkar. 1999. “Population Momentum for Gradual Demographic Transitions,” *Population Studies*, July, 53(2): 255-262.

Li, Nan and Shripad Tuljapurkar. 2000. “The Solution of Time-Dependent Population Models,” *Mathematical Population Studies*, 7(4): 311-329.

O’Neill, Brian C., Sergei Scherbov, and Wolfgang Lutz. 1999. “The Long-Term Effect of the Timing of the Fertility Decline on Population Size,” *Population and Development Review*, December, 25(4): 749-756.

Preston, Samuel H. 1986. “The Relation between Actual and Intrinsic Growth Rates,” *Population Studies*, November, 40(3): 343-351.

Preston, Samuel H. 1988. “Reply to Wachter,” *Population Studies*, November, 42(3): 495-501.

Preston, Samuel H. and Michel Guillot. 1997. "Population Dynamics in an Age of Declining Fertility," *Genus*, 53(3-4): 15-31.

Schoen, Robert and Stefan Hrafn Jonsson. 2003. "Modeling Momentum in Gradual Demographic Transitions," *Demography*, November, 40(4): 621-635.

Schoen, Robert and Young J. Kim. 1991. "Movement Toward Stability as a Fundamental Principle of Population Dynamics," *Demography*, August, 28(3): 455-466.

Schoen, Robert and Young J. Kim. 1998. "Momentum Under a Gradual Approach to Zero Growth," *Population Studies*, November, 52(3): 295-299.

United Nations. 2007. *World Population Prospects: The 2006 Revision, CD-ROM Edition* (United Nations publication, Extended Dataset, Sales No. E.07.XIII.7), New York: United Nations.

Vincent, Paul. 1945. "Potentiel d'Accroissement d'Une Population," *Journal de la Société de Statistique de Paris*, 1re Série, Janvier-Février, 86(1-2): 16-39.

World Health Organization. 2008. *Life Tables for WHO Member States*. http://www.who.int/whosis/database/life_tables/life_tables.cfm. Last accessed January 13, 2009.

Table 1. Total, Weak, and Strong Momentum for the World and Major Regions, 2005

Region	Total Momentum	Weak Momentum	Strong Momentum	Weak x Strong
WORLD	1.398	1.193	1.173	1.399
More developed regions	0.971	1.302	0.745	0.969
Less developed regions	1.437	1.178	1.220	1.438
Least developed countries	1.513	1.033	1.468	1.515
Less developed regions, excluding least developed countries	1.395	1.234	1.131	1.396
Less developed regions, excluding China	1.494	1.131	1.322	1.496
Sub-Saharan Africa	1.463	1.011	1.449	1.465
AFRICA	1.503	1.033	1.458	1.506
ASIA	1.388	1.265	1.098	1.388
EUROPE	0.929	1.383	0.669	0.926
LATIN AMERICA AND THE CARIBBEAN	1.486	1.257	1.182	1.486
NORTHERN AMERICA	1.110	1.157	0.960	1.110
OCEANIA	1.299	1.159	1.120	1.299

Source: Authors' calculations.

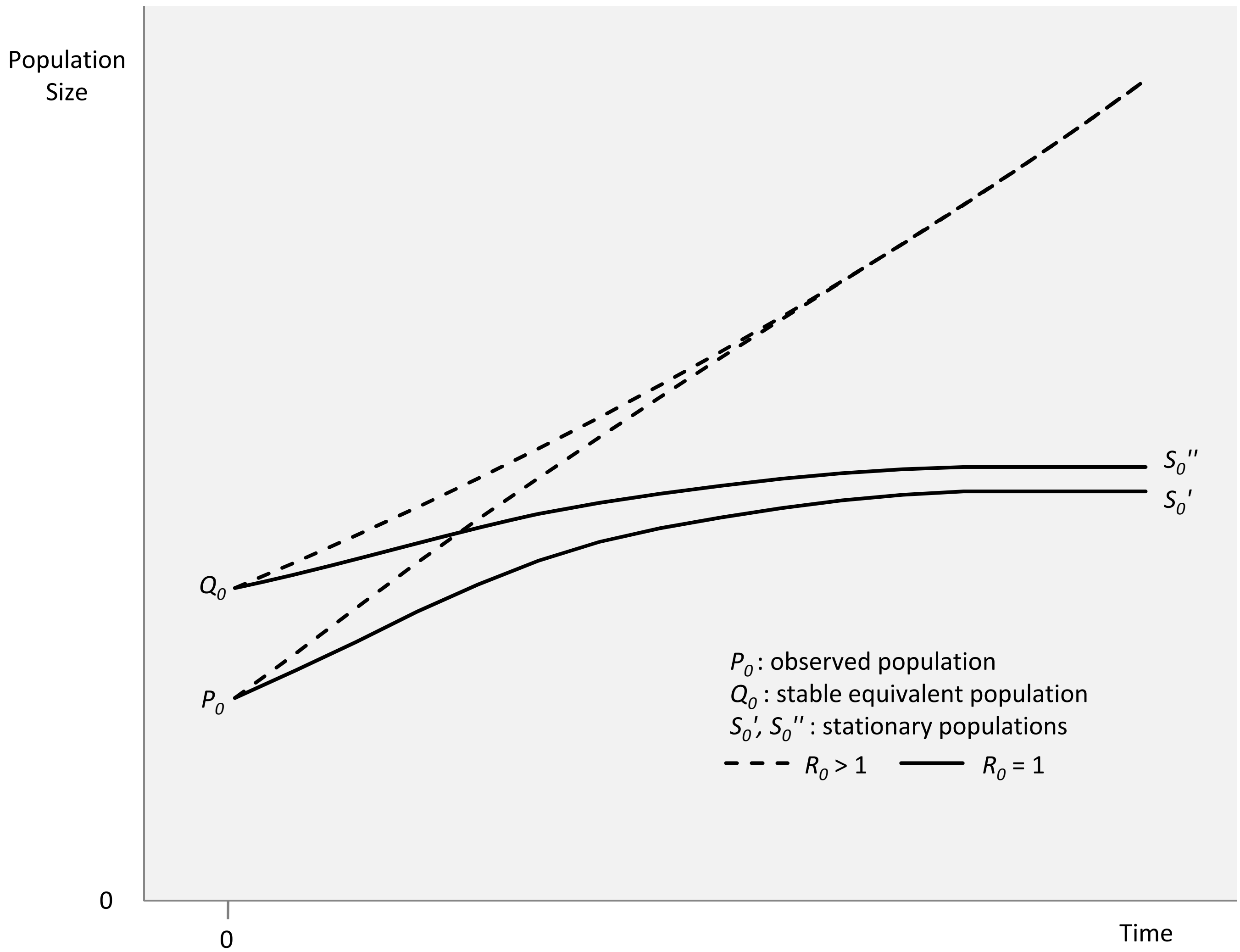

Figure 1. Framework for Understanding Total, Weak, and Strong Population Momentum

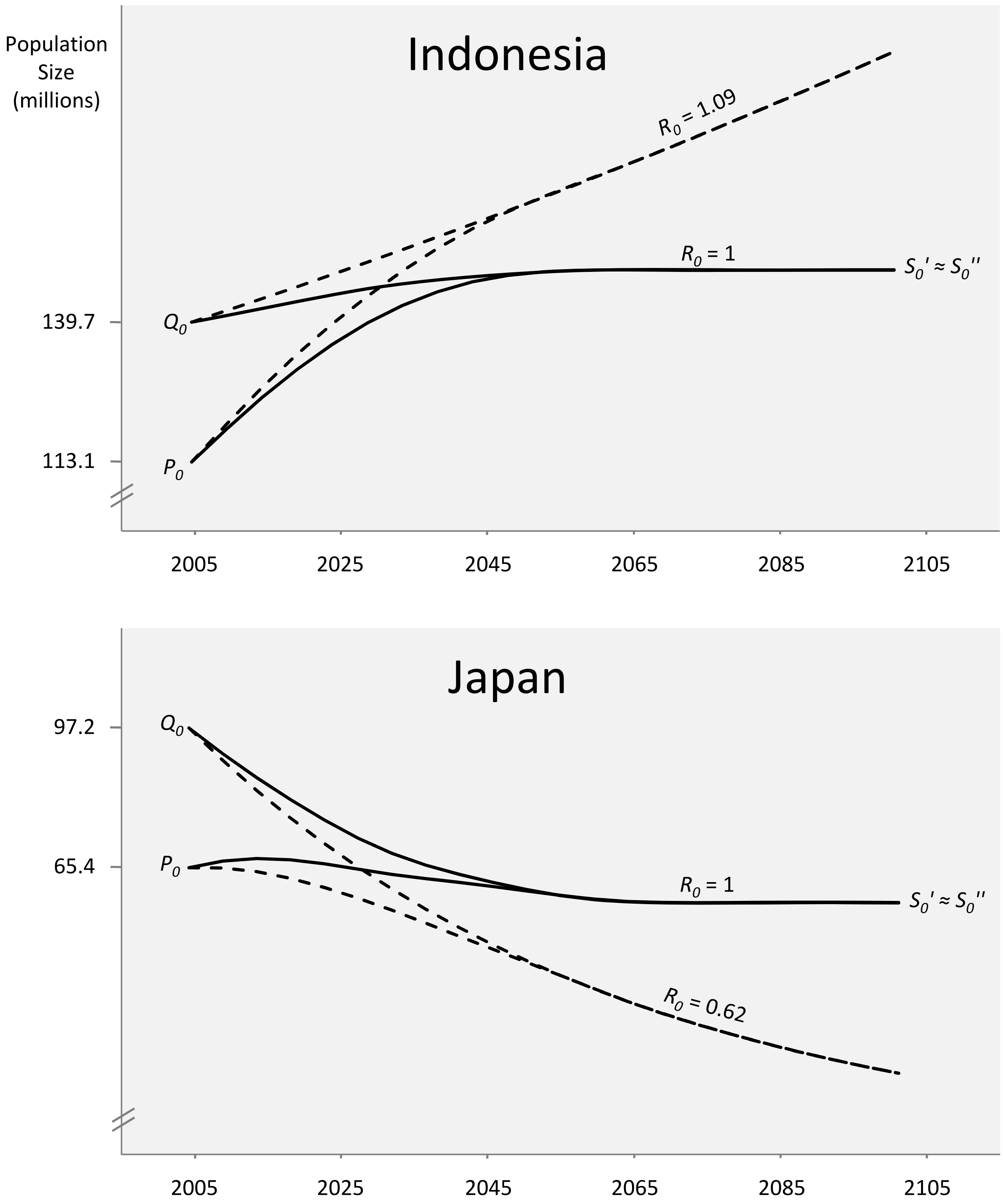

Figure 2. Projections of the Female Populations of Indonesia and Japan, 2005-2105

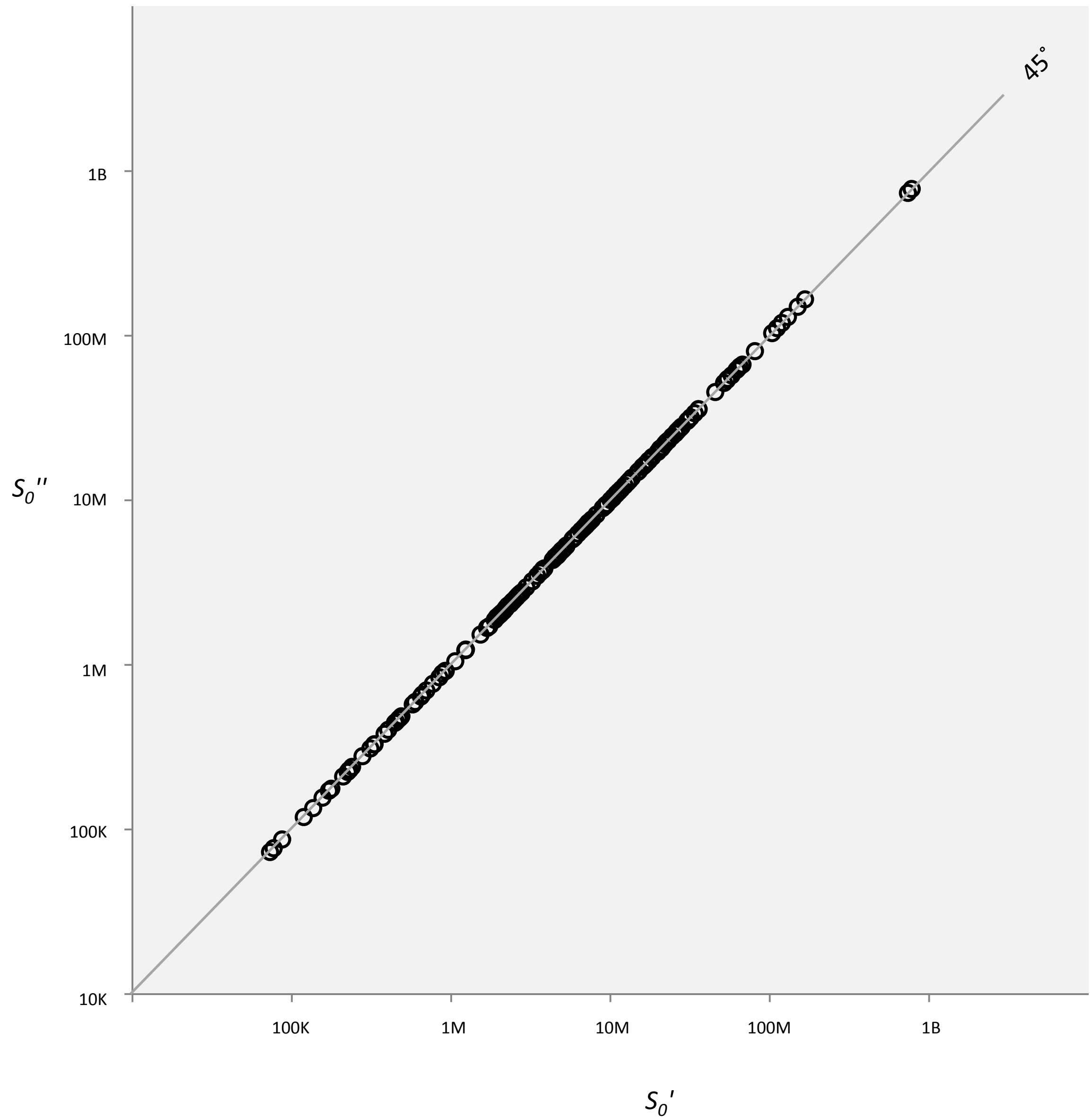

Figure 3. Relationship Between S_0' and S_0'' (N = 176 countries)

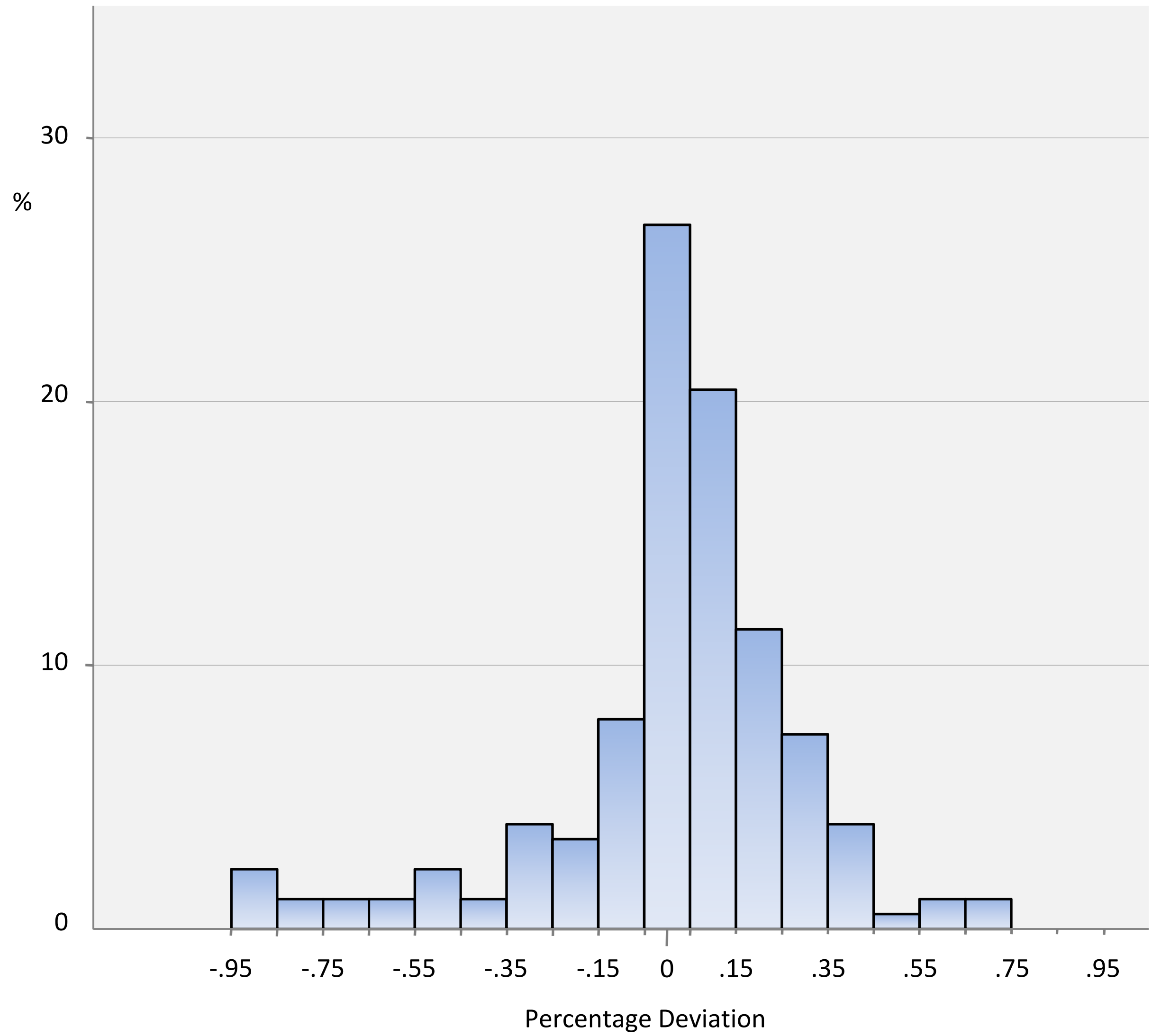

Figure 4. Distribution of Percentage Deviations Between S_0' and S_0''
(N = 176 countries)

Notes :
a - Deviations are calculated as $[(S_0'' - S_0') / S_0'] \times 100$.
b - Five countries fall outside the interval (-0.95, 0.95), ranging from Russia (-1.014) to Eritrea (1.168).

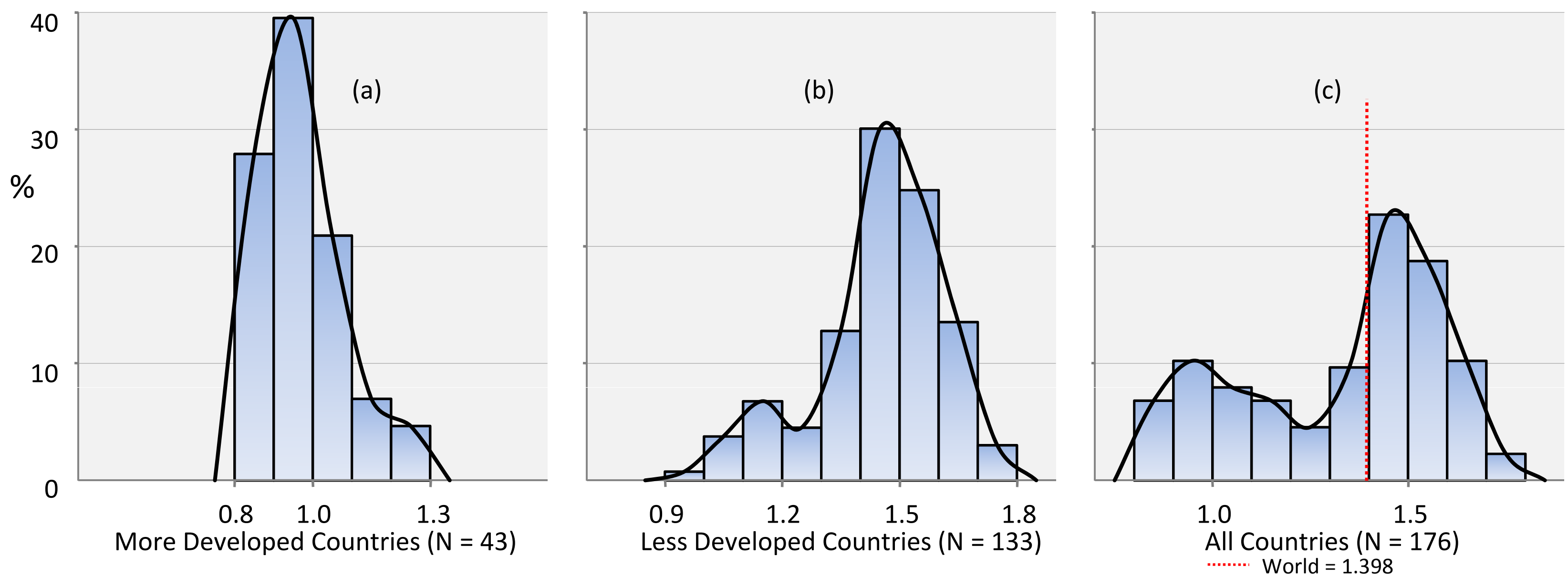

Figure 5. Distribution of Total Momentum by Level of Development, 2005

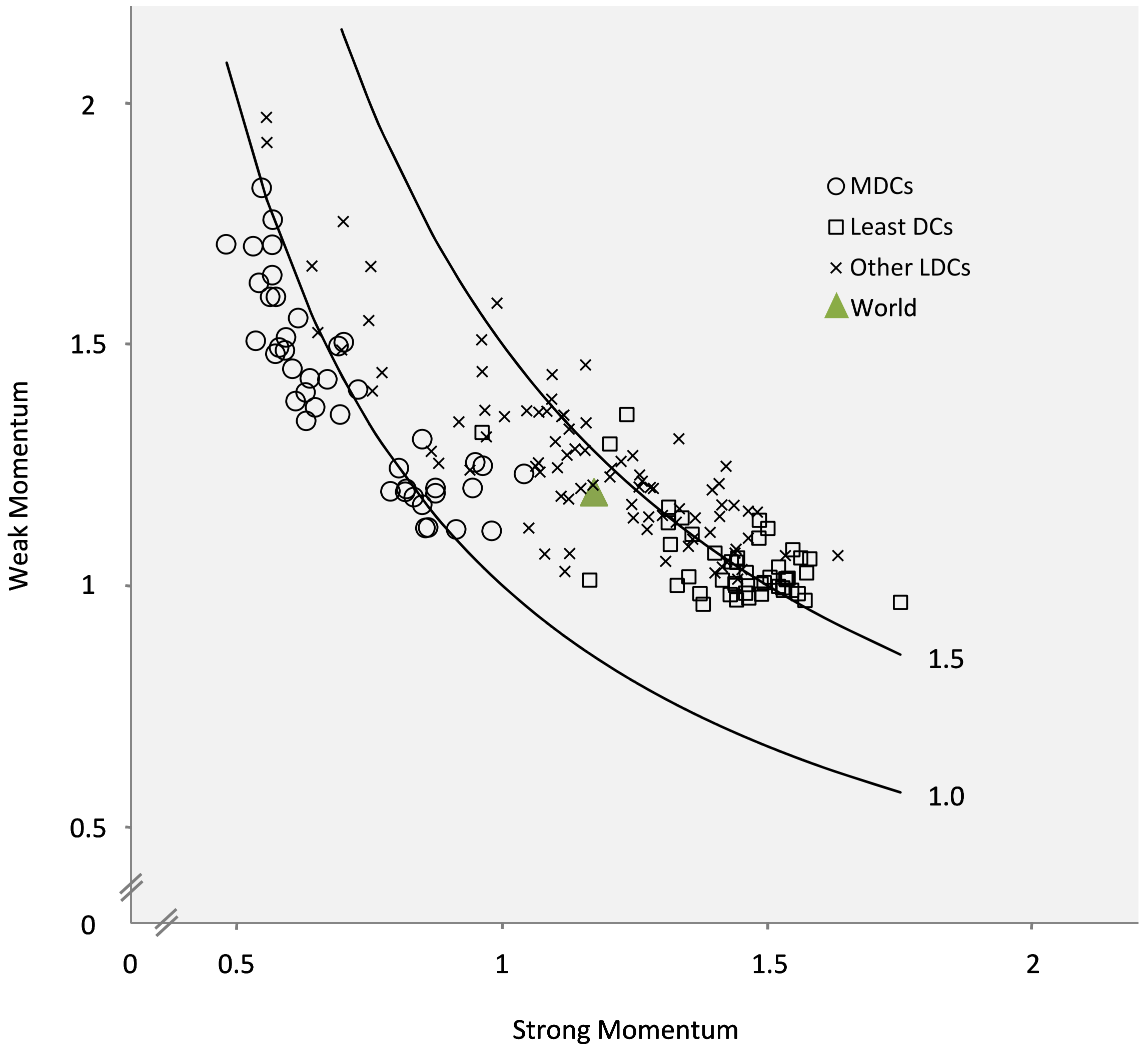

Figure 6. Plot of Weak and Strong Momentum for Individual Countries by Level of Development, 2005 (N = 176 countries)

Note: MDCs: More Developed Countries; LDCs: Less Developed Countries